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Additional Two-Dimensional Wake and Jet-Like Flows

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I. Introduction

A NUMBER of authors[1–4] have described wake and jetlike similarity solutions to the steady laminar, incompressible boundary-layer equations. The purpose of this Note is to extend these solutions into several new areas.

II. Equations

Solutions are sought to the two-dimensional incompressible boundary-layer equations

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

with the general boundary conditions

$$y = 0: \quad u = u_w(x),\ v = 0$$
$$y \to \infty: \quad u \to u_e(x)$$

A convenient form of the similarity equations may be obtained by assuming that

$$G'(\eta) = (u-u_e)/(u_w-u_e), \quad \eta = y/h(x) \tag{3}$$

so that Eqs. (1) and (2) become

$$\left.\begin{aligned} G''' + A[(G+B\eta)G'' - \beta(G'^2+2BG')] = 0 \\ G(0) = G'(\infty) = 0, \quad G'(0) = 1 \end{aligned}\right\} \tag{4}$$

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where

$$A = \pm 1$$
$$B = u_e/(u_w - u_e)$$
$$u_e \sim u_w \sim x^{\beta/(2-\beta)}$$
$$h(x) = [(2-\beta)AB\nu x/u_e]^{1/2}$$

Wake and jetlike solutions to Eq. (4) have the added restriction that $G''(0) = 0$. The present work is intended to consider those values of A, B, and β for which such solutions can be found.

The parameter B essentially defines the wall to edge velocity ratio, and β is the usual velocity gradient parameter. The constant A is a more complicated parameter because it takes on the values of ± 1 depending on both B and β. If the solutions are to be real then h must be real and $(2-\beta)ABx/u_e \geqq 0$. Furthermore, by considering the asymptotic behavior of Eq. (4) at large η, one may show that the outer boundary condition is approached through an exponential decrease (decay) of G' for the case when $AB > 0$ and $\beta \leqq 2$. There are two other cases: first for $\beta > 2$, $AB > 0$ the outer boundary condition is again approached with exponential decay of G' but this is a so called "backward boundary layer"[5] in which $x/u_e < 0$. The second case is where $AB < 0$ with $\beta < -\frac{1}{2}$ and $x/u_e < 0$ and thus this is also a backward boundary layer but with asymptotic algebraic decay of G'. The latter case is not the usual kind of boundary layer although it has been discussed elsewhere[5,6] in some detail.

III. Discussion

Stewartson's[1] original work and the more recent paper by Kennedy[2] concerning the wakelike flows in the region $B < 0$ is shown in Fig. 1. Note that $B = -1$ and $\beta = -0.1988$ specifies the Falkner-Skan separation profile, and at $B = -1$ and $\beta = 0$ the solution corresponds to Chapman's free shear layer solution.[2] Steiger and Chen[3] have extended these results to $B > 0$ which are jet-like solutions (u_e may be considered positive without loss of generality; thus $B \geqq 0$ corresponds to $u_e \leqq u_w$ and these flows may be designated as jets). They did not point out however that the limiting case of $u_e = 0$ (i.e., $B = 0$) corresponds to Schlichting's[4] two-dimensional jet into a quiescent fluid. Steiger and Chen also reported the counterflow jet solutions where $-\frac{1}{2} \leqq B \leqq -\frac{1}{3}$ and $\beta \geqq 1$. In a recent investigation,[7] no solutions with $G''(0) = 0$ were found in the $0 < \beta < 1$ range which confirms Steiger and Chen's observation.

Limit $\beta \to \pm\infty$

Steiger and Chen[3] reported that when $\beta = \infty$, $u_w/u_e = -2$ (i.e., $B = -\frac{1}{3}$), based on numerical integration of the similarity

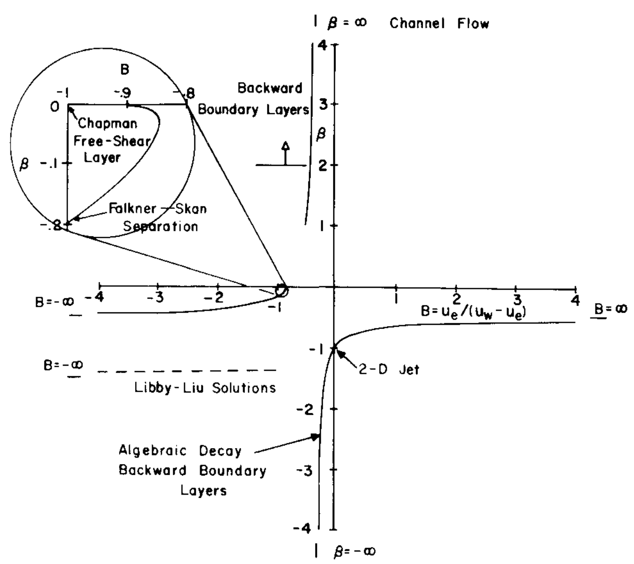

Fig. 1 $B-\beta$ map of wake and jetlike solutions.

equations. A closed-form solution can be obtained if Eq. (4) is treated as a singular perturbation problem where the limiting form of the equation is obtained through the use of the following transformations:

$$\bar{G} = (|\beta|)^{1/2} G \quad \text{and} \quad \bar{\eta} = (|\beta|)^{1/2}\eta$$

Dividing the resulting equation by β and taking the limit for large β gives

$$\bar{G}''' - A\sigma(\bar{G}'^2 + 2B\bar{G}') = 0$$

where $\sigma = \beta/|\beta|$. This equation can be integrated twice in closed form using the boundary conditions that as $\bar{\eta} \to \infty$, $\bar{G}' \to 0$, $\bar{G}'' \to 0$, and $\bar{\eta} = 0$, $\bar{G}' = 1$ to give (note that $AB\sigma = |B|$):

$$\bar{G}' = 3B\llbracket \tanh^2\{(|B|/2)^{1/2}\eta + \tanh^{-1}[(\tfrac{1}{3}B+1)]^{1/2}\} - 1\rrbracket \quad (5)$$

where $B \leqq -\frac{1}{3}$ and $\bar{G}' > 0$. Equation (5) is a generalization of the converging channel flow problem[8] ($B = -1$) to include the moving wall situation. The $B = -\frac{1}{3}$ limit agrees with the results of Steiger and Chen and has the following velocity distribution

$$\bar{G}' = 1 - \tanh^2[\bar{\eta}/(6)^{1/2}]$$

Algebraic Decay Solutions

The limiting cases at $B = -\frac{1}{3}$, $\beta = -\infty$, and $B = 0$, $\beta = -1$ are connected by a set of solutions involving asymptotic algebraic decay of G' with $\bar{\eta}$. The asymptotic form of G' at large η with $AB < 0$ and $\beta < 0$ can be written as (following techniques discussed in Ref. 9):

$$G'(\eta) \sim [(|AB|)^{1/2}(\eta - \delta^*/B)]^{2\beta}$$

These solutions have infinite displacement thicknesses for $\beta \geqq -\frac{1}{2}$ but are well behaved for smaller β. For $-1 > \beta > -\frac{1}{2}$ the solutions are difficult to numerically compute because of the large algebraic tail. Solutions were obtained at $\beta = -1.1$, -1.5, and -2.0 over a range of B and were used to determine the value of B for $G''(0) = 0$ by interpolation; the results are shown in Fig. 1.

Limit $u_w = u_e$

The fact that the wakelike solutions ($B < -1$) or the jetlike solutions ($B > 0$) have a limiting point at $B = \pm\infty$ and $\beta = -\frac{1}{2}$ has been recognized by other investigators. At that point $u_w = u_e$ and thus $u \equiv u_e$ throughout the boundary layer so long as G' is finite. Although the velocity distributions are degenerate, the limiting case may have some interest. If the limiting case is considered as a singular perturbation problem where the singularity is removed by the transformations

$$G(\eta) = F(\eta_1)/(|AB|)^{1/2} \quad \text{and} \quad \eta = \eta_1/(|AB|)^{1/2}$$

one finds that the equation for $F(\eta_1)$ is identical to the large η_1 approximation, i.e.,

$$F''' + \eta_1 F'' - 2\beta F' = 0, \quad F'(\eta_1) \equiv dF/d\eta_1 \quad (6)$$

A series solution of the following form can be assumed:

$$F' = K\eta_1\gamma \exp(m\eta_1^2) \sum_{n=1}^{\infty} p_n/\eta_1^n$$

The result, which has been derived in Ref. 7, has special properties at certain values of β given by

$$\beta_n = -(2n-1)/2, \quad n = 1, 2, 3, \ldots$$

The series then reduces to a finite number of terms and can be made to satisfy the inner as well as outer boundary conditions. Some of these solutions are given in Table 1.

Table 1 Limiting solutions for $F''(0) = 0$

n	β_n	$F'(\eta_1)$
1	$-\frac{1}{2}$	$\exp(-\eta_1^2/2)$
2	$-\frac{3}{2}$	$(1-\eta_1^2)\exp(-\eta_1^2/2)$
3	$-\frac{5}{2}$	$(1-2\eta_1^2+\eta_1^4/3)\exp(-\eta_1^2/2)$
...	...	...
...	...	...

The $n = 2$ case corresponds to the limiting case for a Libby-Liu[10] family of solutions with boundary-layer velocity overshoot. The $G''(0) = 0$ solutions for this family are indicated as a dashed line in Fig. 1 because the $B = -1$ point is the only other value currently available. Higher n's correspond to the larger negative β families whose existence was noted by Libby-Liu.[10] Solutions to Eq. (6) may be obtained for any β by numerical integration; however, $G''(0)$ is infinite for all β (except at β_n) because $G''(0) = (|AB|)^{1/2}F''(0)$. These results for the limiting case may be used to determine the large B approximations in a perturbation series approach similar to that used by Mirels[11] for $\beta = 0$; one has

$$G(\eta, B) = [1/(|B|)^{1/2}]F[(|B|)^{1/2}\eta] + 0(1/|B|)$$

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Inverse Relationships in Equilibrium Statistical Thermodynamics

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I. Introduction

EMPIRICAL (sometimes called classical or phenomenological) thermodynamics provides relationships among physical quantities. Statistical thermodynamics provides methods of calculating the physical quantities within the framework of specified atomic models. Such relationships may be called *direct relationships*. It is usually rather difficult to go in the opposite direction, namely to determine a specific model from a series of measured values of some physical quantity. The

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